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A SIMPLIFIED NONLINEAR THEORY OF THE GENERALIZED PLANE STRAIN

MASANAO SHIBUI
Department of Mechanical Engineering, Tohoku University, Sendai, Japan

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Abstract—A finite deformation theory of plane strain is formulated for transversely isotropic, homogeneous bodies with nonlinear stress–strain law. A new set of simplified field equations, which is valid in the case of some deviations from Hooke's law, is derived systematically with the help of the method of order estimation. For illustration purposes, a circular hole in a body under generalized plane strain is considered, together with the solution of an example problem by perturbation techniques.

1. INTRODUCTION

In connection with the design of structural members, finite or large deformation theory based on nonlinear continuum mechanics has played an important role in the determination of the stress field for the range of large deformations. Because it is often difficult to find an explicit solution of a given three-dimensional problem, there are many papers for the studies of elastic or plastic bodies under their analogues in two dimensions[1–4]. However, most of these studies are confined to the treatments based on the physically nonlinear and geometrically linear theory.

Green and Adkins[5] dealt with the large deformation theory of elasticity. Their theory is valid for geometrically nonlinear ranges, but it is difficult to apply their theory directly to the problem of general compressible materials.

In most of these works, however, materials are assumed to be isotropic. A number of materials of interest to engineers exhibit nonlinear mechanical effects, even when sustaining small deformations. These effects are frequently thought to be due to anisotropic properties of materials. In order to get a better understanding of the actual phenomena, not only physical and geometrical nonlinearities but also anisotropy of material should be taken into account.

When the deformation is infinitesimal, i.e. when the nondimensional stress components are sufficiently small, the linear stress–strain law is valid with good accuracy. If the nondimensional stress becomes relatively large, the second and the higher order terms can not generally be neglected. In the study based on the physically nonlinear and geometrically linear theory, the nonlinear terms are taken into account in the stress–strain law, but the corresponding nonlinear terms are omitted in the other field equations. Such a theory may be valid if the deformations are very small and also if the values of the coefficients in the stress–strain law are suitably chosen, but it does not necessarily give satisfactory results for materials with arbitrary nonlinear stress–strain law. Accordingly the general finite deformation theory is to be required.

The present investigation is concerned with the simplified theory for homogeneous, transversely isotropic bodies under generalized plane strain in the case of some deviations from Hooke's law for the range of finite deformations. In this paper, however, we confine our attention to the case in which the deformation is smooth. First, the definition for the generalized nonlinear plane strain is proposed which is consistent with the usual infinitesimal theory and also with the three-dimensional finite deformation theory. Any simplified nonlinear theory of continuum mechanics is necessarily an approximation from the general finite deformation theory. In the process of simplification of the theory it may be unreasonable to make intuitive choice of several terms which constitute the basic equations. Possible systematic formulations should be employed in deriving the simplified theory. Here the method of the simplification employed in deriving the nonlinear thin shell theory[6] is applied to the finite deformation theory of the generalized plane strain problem. The maximum for the magnitude of the strain is denoted by $o(\gamma)$. A consistent approximation with the general theory is made by taking all terms larger than $o(\gamma^3)$ in the compatibility equations. The differential equations obtained are simple but different from those by the previous authors[1–4] in the sense that stress functions defined for the in-plane stresses are coupled with a stress function for the anti-plane stresses.

Secondly, a circular hole in a body subjected to both uniform tension and uniform longitudinal shear is considered for illustration purposes. It is true that nonlinear material constants which constitute the nonlinear terms in the nonlinear stress–strain relations should be determined from tension and shear tests in the manner of Orthwein[7], but these constants are determined under some reasonable suppositions which are consistent with the problem considered. These suppositions are employed because a set of appropriate experimental data is not at hand.

In the definition and relations of Sections 2–8, the Latin indices range over 1–3, whereas the Greek indices have the range 1, 2 unless stated otherwise.

2. DEFINITION OF THE GENERALIZED PLANE STRAIN

We consider a body to undergo a deformation in which a point initially at X^i referred to fixed rectangular cartesian coordinates moves to x^i in the same coordinates. The convected curvilinear coordinate system ξ^i is chosen so that $\xi^i = X^1$.

The metric tensor of the undeformed body is given by

$$A_{ij} = A^{ij} = \delta_j^i \tag{2.1}$$

where $\delta^i_{.j}$ denotes the Kronecker delta.

In the following the surface $\xi^3 =$ constant shall be called the plane, and the stresses on that plane shall be simply called the in-plane stresses.

It is convenient to introduce the following stress tensors[11]

$$T^i = s^{rm} N_r x^i_{,m} = t^{rm} N_r X^i_{,m} \tag{2.2}$$

where the comma denotes the partial differentiation with respect to X^i, and T^i is the stress vector, per unit area of undeformed body, associated with a surface in the deformed body, whose unit normal in its undeformed position is $N^r X^i_{,r}$.

In the following the symbol "o" will be used in the conventional sense except that dependence on the complementary energy function W is permitted[8]. That is, the relation $A = o(B)$, where $B \geqq 0$, means that for a given complementary energy function W there exists a positive number K such that $|A| \leqq KB$.

We denote the maximum "length" of the strain measure γ_{ij} by $\gamma = \max(\sqrt{\gamma^i_{.j}\gamma^j_{.i}})$†, where $\gamma^i_{.j} = \gamma_{kj} A^{ki}$.

For convenience of proceeding with an argument, we introduce a mixed tensor defined by

$$\bar{s}^i_{.j} = s^{ik} A_{kj}. \tag{2.3}$$

†$\gamma^i_{.j}\gamma^j_{.i} \geqq 0$ for $\gamma^i_{.j}$ real.

It should be noted that the mixed tensor $\bar{s}^i_{\cdot j}$ is discriminated from $s^i_{\cdot j} = s^{ik}a_{kj}$, where a_{kj} is the covariant metric tensor of the deformed body.

DEFINITION: A body is said to be in the state of the generalized plane strain if it deforms by the application of external forces such that the deformation is independent of ξ^3 within the error which is of $o(\gamma^3)$ at most, and if the complementary energy function W satisfies

$$\|\bar{W}(\bar{s}^i_{\cdot j})\| = 0(\|\bar{s}^i_{\cdot j}\|^2) \tag{2.4a}$$

$$\left\|\frac{\partial \bar{W}}{\partial \bar{s}^i_{\cdot j}}\right\| = 0(\|\bar{s}^i_{\cdot j}\|) \tag{2.4b}$$

where the symbol "$\|\bar{s}^i_{\cdot j}\|$" denotes the L_2 norm of the function $\bar{s}^i_{\cdot j}$.

The requirements imposed on the definition of the generalized plane strain are more general than those proposed by Green and Adkins[5] in the sense that the displacement component such as V^3 is not necessarily neglected in the definition of the plane strain. It is worth while noting that the requirement such that $V^3 \neq 0$ is more consistent with the general finite deformation theory.

By the definition the displacement can be written in the form;

$$x^k = X^k + \bar{V}^k(X^1, X^2) + o(\gamma^3) \tag{2.5a}$$

$$\bar{V}^k, 3 = 0(\gamma^3). \tag{2.5b}$$

The strain tensor γ_{ij} in the state of the generalized plane is found from (2.5) and from the definition of the generalized plane strain as follows:

$$\gamma_{\alpha\beta} = \frac{1}{2}(\bar{V}_{\alpha,\beta} + \bar{V}_{\beta,\alpha} + \bar{V}^k_{\cdot\alpha}\bar{V}_{k,\beta}) + o(\gamma^3) \tag{2.6a}$$

$$\gamma_{\alpha 3} = \frac{1}{2}\bar{V}_{3,\alpha} + o(\gamma^3) \tag{2.6b}$$

$$\gamma_{33} = o(\gamma^3). \tag{2.6c}$$

Now it is assumed that the displacement V^i and its first and second derivatives may be finite but their least upper bound is fairly small compared with unity. A consistent approximation with the general finite deformation theory is made by taking all terms larger than $o(\gamma^3)$ in the final differential equations.

3. COMPATIBILITY EQUATIONS

We shall state the compatibility equation for the generalized plane strain problem. This equation can be obtained from the condition that the Riemann–Christoffel tensor r_{ijkl} of the deformed body vanishes identically. In our coordinate system the condition $r_{ijkl} = 0$ gives the following relation between the strain components[12]:

$$J^{(ij)} = \frac{1}{2}(\epsilon^{imn}\epsilon^{jpq} + \epsilon^{jmn}\epsilon^{ipq})\left(\gamma_{qn,mp} - \frac{1}{2}N_{mnpq}\right) = 0 \tag{3.1}$$

where ϵ^{imn} is the three dimensional permutation tensor. Here the tensor N_{mnpq} is given by

$$N_{mnpq} = T_{mqs} T_{npt} a^{st} \tag{3.2}$$

$$T_{mqs} = \gamma_{\{mq,s\}} \tag{3.3}$$

where we have introduced the notations such that

$$\gamma_{\{mq,s\}} = -\gamma_{mq,s} + \gamma_{qs,m} + \gamma_{sm,q} \tag{3.4a}$$

$$J^{(ij)} = \frac{1}{2}(J^{ij} + J^{ji}) \tag{3.4b}$$

and a^{st} is the contravariant metric tensor of the deformed body. By the definition of the generalized plane strain (3.1) is rewritten in the form:

$$J_{(\alpha\beta)} = -T_{\kappa 3\lambda} T_{\kappa 3\lambda} \delta_{\alpha\beta} + T_{\alpha 3\lambda} T_{\beta 3\lambda} + o(\gamma^3) \tag{3.5a}$$

$$J_{(\alpha 3)} = T_{\alpha 3\rho,\rho} - T_{\alpha 3\lambda} T_{\rho\rho\lambda} + T_{\rho 3\lambda} T_{\rho\alpha\lambda} + o(\gamma^3) \tag{3.5b}$$

$$J_{(33)} = \gamma_{\rho\rho,\alpha\alpha} - \gamma_{\alpha\rho,\alpha\rho} + \frac{1}{2}(\gamma_{\alpha\alpha,\lambda}\gamma_{\rho\rho,\lambda} - 4\gamma_{\alpha\alpha,\lambda}\gamma_{\rho\lambda,\rho} + 4\gamma_{\alpha\lambda,\alpha}\gamma_{\rho\lambda,\rho} + 2\gamma_{\alpha\rho,\lambda}\gamma_{\alpha\lambda,\rho} - 3\gamma_{\alpha\rho,\lambda}\gamma_{\alpha\rho,\lambda})$$
$$+ 2\gamma_{3\alpha,\alpha}\gamma_{3\rho,\rho} - \gamma_{3\alpha,\rho}\gamma_{3\alpha,\rho} - \gamma_{3\alpha,\rho}\gamma_{3\rho,\alpha} + o(\gamma^3) \tag{3.5c}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta.

It is easily found from (3.5) that (3.1) is reduced to

$$\epsilon^{\alpha\rho}\gamma_{\alpha 3,\rho} = o(\gamma^3) \tag{3.6}$$

$$J^{(33)} = o(\gamma^3) \tag{3.7}$$

where $\epsilon^{\alpha\rho}$ is the two-dimensional permutation tensor.

Equations (3.6) and (3.7) provide a set of compatibility equations for the body under generalized plane strain.

4. CONSTITUTIVE EQUATIONS

For an elastic material which is homogeneous and transversely isotropic in the constrained state, the constitutive equations can be derived by [10]

$$\gamma^i_{.j} = \frac{1}{2}\left(\frac{\partial \bar{W}}{\partial \bar{s}^{\,j}_{.i}} + \frac{\partial \bar{W}}{\partial \bar{s}^{\,i}_{j}}\right) \tag{4.1}$$

where W is a complementary energy function expressed in the form:

$$\bar{W} = \bar{W}(s_1, s_2, s_3, s_4, s_5). \tag{4.2}$$

Here s_i, $i = 1, 2, \ldots, 5$ are the five stress invariants for the transverse isotropy with respect to a direction which is parallel to the X^3-axis. In our coordinate system they are given by

$$s_1 = \bar{s}^{\,i}_{.i},\ s_2 = \bar{s}^{\,i}_{.j}\bar{s}^{\,j}_{.i},\ s_3 = \bar{s}^{\,i}_{.j}\bar{s}^{\,j}_{.k}\bar{s}^{\,k}_{.i},\ s_4 = \bar{s}^{\,3}_{.3},\ s_5 = \bar{s}^{\,\alpha}_{.3}\bar{s}^{\,3}_{.\alpha}. \tag{4.3}$$

We restrict our attention to the compressible material. Following the same notations as John [8], we can write

$$\bar{s}^i_{.j} = o(s) = O(\gamma) \tag{4.4}$$

where s is the maximum "length" of the stress tensor $\bar{s}^i_{.j}$ defined by

$$s^2 = \max_{x^i} (\bar{s}^i_{.j}\bar{s}^j_{.i}). \tag{4.5}$$

For monotonically increasing loads, at first the linear stress–strain relation exists, then with continuously increasing loads the deformation gradually changes. It can be assumed that the complementary energy function is an analytic function of these stress invariants in the neighbourhood of $s_i = 0$, $i = 1,2,\ldots,5$, and therefore it can be expanded in the form:

$$\begin{aligned}\bar{W} = {} & a_1 s_1 + a_2 s_2 + a_3 s_3 + a_4 s_4 + a_5 s_5 + a_6 s_1^2 + a_7 s_2^2 + a_8 s_4^2 + a_9 s_5^2 + a_{10} s_1 s_2 + a_{11} s_1 s_3 + a_{12} s_1 s_4 \\ & + a_{13} s_1 s_5 + a_{14} s_2 s_4 + a_{15} s_2 s_5 + a_{16} s_3 s_4 + a_{17} s_4 s_5 + a_{18} s_1^3 + a_{19} s_4^3 + a_{20} s_1^2 s_2 + a_{21} s_1^2 s_4 + a_{22} s_1^2 s_5 \\ & + a_{23} s_1 s_4^2 + a_{24} s_2 s_4^2 + a_{25} s_4^2 s_5 + a_{26} s_1^4 + a_{27} s_4^4 + a_{28} s_1^2 s_4^2 + a_{29} s_1^3 s_4 + a_{30} s_1 s_4^3 + o(s^5)\end{aligned} \tag{4.6}$$

where a_2, a_5, a_6, a_8 and a_{12} denote the linear material constants, whereas the others mean the nonlinear mater'al constants.

If we restrict our attention to a body which is unstressed before deformation, it is found that $a_1 = a_4 = 0$.

Substituting the series (4.6) into (4.1), we obtain

$$\begin{aligned}\gamma^{\alpha}_{.\beta} = {} & (2a_6 s_1 + a_{10} s_2 + a_{12} s_4 + a_{13} s_5 + 3a_{18} s_1^2 + 2a_{21} s_1 s_4 + a_{23} s_4^2)\delta_\beta^{\alpha} + 2\bar{s}_\beta^{\alpha}(a_2 + a_{10} s_1 + a_{14} s_4) \\ & + 3a_3 \bar{s}_m^{\alpha} \bar{s}_\beta^{m} + o(\gamma^3)\end{aligned} \tag{4.7a}$$

$$\gamma^{\alpha}_{.3} = (2a + a_5)\bar{s}_3^{\alpha} + \{2a_{10} + a_{13})s_1 + (2a_{14} + a_{17})s_4\}\bar{s}_3^{\alpha} + 3a_3 \bar{s}_m^{\alpha} \bar{s}_3^{m} + o(\gamma^3) \tag{4.7b}$$

$$\begin{aligned}\gamma^{3}_{.3} = {} & (2a_6 + a_{12})s_1 + (a_{12} + 2a_2 + 2a_8)s_4 + (a_{10} + a_{14})s_2 + (a_{13} + 3a_3 + a_{17})s_5 + (3a_{18} + a_{21})s_1^2 \\ & + (2a_{21} + 2a_{10} + 2a_{23})s_1 s_4 + (a_{23} + 2a_{14} + 3a_3 + 3a_{19})s_4^2 + o(\gamma^3).\end{aligned} \tag{4.7c}$$

For the sake of compactness let us put

$$\begin{aligned} & D_1 = 2(a_2 + a_6),\ D_2 = 2a_6,\ D_3 = 2a_6 + a_{12},\ D_4 = 2(a_2 + a_6 + a_8 + a_{12}), \\ & D_5 = 2a_2,\ D_6 = 2a_2 + a_5,\ A = D_2 - \frac{D_3^2}{D_4}.\end{aligned} \tag{4.8}$$

Equations given by (4.7) are the constitutive equations for the generalized plane strain based on the finite deformation theory. They contain 14 constants.

5. STRESS FUNCTIONS

It is easily found from (2.1) that stress tensor t^{ij} and s^{ij} are connected by the relation [11]

$$t^{ij} = s^{ir}(\delta_r^j + \bar{V}^j_{.r}). \tag{5.1}$$

The equations of equilibrium for the exact three-dimensional theory in the absence of body forces are given by

$$t_i^{ij} = 0. \tag{5.2}$$

By the definition of the generalized plane strain, (5.2) can be written in the form:

$$t_{,\alpha}^{\alpha 3} = o(s^3) \tag{5.3a}$$

$$t_{,\alpha}^{\alpha\beta} = o(s^3). \tag{5.3b}$$

Here we introduce the stress functions F, G and Φ defined by

$$t^{\alpha\beta} = \epsilon^{\alpha\rho}\epsilon^{\beta\lambda}F_{,\rho\lambda} + \epsilon^{\alpha\rho}G_{,\rho}^{\beta} + o(s^3) \tag{5.4a}$$

$$t^{\alpha 3} = \epsilon^{\alpha\rho}\Phi_{,\rho} + o(s^3). \tag{5.4b}$$

By the introduction of these stress functions (5.3) is satisfied identically within the error which is of $o(s^3)$ at most.

When the deformation is very small, the linear theory is valid. If we neglect the rigid body motion, the in-plane strain components $\gamma_{\alpha\beta}$ of order $o(\gamma)$ and the in-plane rotation $\omega_{\alpha\beta}$ of order $o(\gamma)$ are expressed in terms of the stress function F as follows:

$$\gamma_{\alpha\beta} = AF_{,\rho\rho}\delta_{\alpha\beta} + D_5\epsilon_{\alpha\rho}\epsilon_{\beta\lambda}F_{,\rho\lambda} + o(\gamma^2) \tag{5.5a}$$

$$\bar{V}_{[\alpha,\beta]} = \omega_{\alpha\beta} = \epsilon_{\alpha\beta}f + o(\gamma^2) \tag{5.5b}$$

where

$$f = -(A + D_5)Q \tag{5.6}$$

$$\bar{V}_{[\alpha,\beta]} = \frac{1}{2}[\bar{V}_{\alpha,\beta} - \bar{V}_{\beta,\alpha}] \tag{5.7}$$

and where Q is a conjugate function with $F_{,\rho\rho}$ in the sense of Cauchy–Riemann. If the function $F_{,\rho\rho}$ is known, its conjugate Q is easily determined except for the constant.

From (5.5) we obtain

$$\bar{V}_{\alpha,\beta} = AF_{,\rho\rho}\delta_{\alpha\beta} + D_5F_{,\rho\rho}\delta_{\alpha\beta} - D_5F_{,\alpha\beta} + \epsilon_{\alpha\beta}f + o(\gamma^2). \tag{5.8}$$

With the aid of (5.1), (5.4), (5.8), stress components s^{ij} are given by

$$\bar{s}_3^{\alpha} = \epsilon^{\alpha\rho}\Phi_{,\rho} - 2D_6\epsilon^{\alpha\lambda}\Phi_{,\lambda}F_{,\rho\rho} + 2D_6\epsilon^{\lambda\kappa}\Phi_{,\kappa}F_{,\lambda}^{\alpha} + o(s^3) \tag{5.9a}$$

$$s^{\alpha\beta} = F_{,\rho}^{\rho}\delta^{\alpha\beta} - F_{\cdot}^{\alpha\beta} - AF_{,\lambda}^{\lambda}F_{,\rho}^{\rho}\delta^{\alpha\beta} + AF_{,\rho}^{\rho}F_{\cdot}^{\alpha\beta} + \epsilon^{\alpha\rho}G_{,\rho}^{\beta} + D_5F_{\cdot}^{\alpha\rho}F_{,\rho}^{\beta} - D_5F_{,\rho}^{\kappa}F_{,\kappa}^{\rho}\delta^{\alpha\beta} + \epsilon^{\alpha\kappa}F_{,\kappa}^{\beta}f + o(s^3). \tag{5.9b}$$

Since s^{ij} is a symmetric tensor, it follows from (5.9b) that

$$\epsilon^{\alpha\beta}(\epsilon_{\alpha\rho}G_{,\beta}^{\rho} + \epsilon_{\alpha\kappa}F_{,\beta}^{\kappa}f) = o(\gamma^3). \tag{5.10}$$

When we determine the stress field of the generalized plane strain, (5.10) becomes an auxiliary equation.

From (2.6c), (4.7c), the stress component $\bar{s}^3_{.3}$ is obtained as follows:

$$\bar{s}^3_{.3} = c_1\bar{s}^{\alpha}_{.\alpha} + c_2\bar{s}^{\alpha}_{.\rho}\bar{s}^{\rho}_{.\alpha} + c_3\bar{s}^3_{.3}\bar{s}^{\alpha}_{.\alpha} + c_4\bar{s}^{\rho}_{.\rho}\bar{s}^{\lambda}_{.\lambda} + o(s^3) \tag{5.11}$$

where

$$c_1 = -\frac{A_1}{A_1 + A_2},\ c_2 = -\frac{A_3}{A_1 + A_2},\ c_3 = -\frac{A_4 + 2A_3}{A_1 + A_2},$$

$$c_4 = \frac{1}{A_1 + A_2}\left\{-A_5 + \frac{A_1(A_6 + 2A_5)}{A_1 + A_2} - \frac{(A_3 + A_5 + A_6 + A_7)A_1^2}{(A_1 + A_2)^2}\right\} \tag{5.12a}$$

where

$$A_1 = 2a_6 + a_{12},\ A_2 = 2a_2 + 2a_8 + a_{12},\ A_3 = a_{10} + a_{14},\ A_4 = 3a_3 + a_{13} + a_{17},$$
$$A_5 = 3a_{18} + a_{21},\ A_6 = 2(a_{10} + a_{21} + a_{23}),\ A_7 = 3a_3 + 2a_{14} + 3a_{19} + a_{23}. \tag{5.12b}$$

Substituting (5.9) into (5.11), we obtain

$$\bar{s}^3_{.3} = c_1F^{\rho}_{,\rho} + (c_4 - Ac_1)F^{\lambda}_{,\lambda}F^{\rho}_{,\rho} + (c_2 - c_1D_5)F^{\kappa}_{,\rho}F^{\rho}_{,\kappa} + c_3\Phi_{,\rho}\Phi^{\rho}_{,} + o(s^3). \tag{5.13}$$

With the aid of (5.9), equations (4.7) can be replaced by

$$\gamma_{\alpha\beta} = (B_1F^{\rho}_{,\rho} + B_2F^{\rho}_{,\rho}F^{\kappa}_{,\kappa} + B_3F^{\lambda}_{,\rho}F^{\rho}_{,\lambda} + B_4\Phi_{,\rho}\Phi^{\rho}_{,})\delta_{\alpha\beta} + B_6F^{\rho}_{,\rho}F_{,\alpha\beta} - B_5F_{,\alpha\beta}$$
$$+ B_7F^{\lambda}_{,\alpha}F_{,\lambda\beta} + B_5\epsilon_{\alpha\rho}G^{\rho}_{,\beta} + B_5\epsilon_{\alpha\kappa}F^{\kappa}_{,\beta}f - B_8\Phi_{,\alpha}\Phi_{,\beta} + o(\gamma^3) \tag{5.14a}$$

$$\gamma_{\alpha 3} = D_6\epsilon^{\alpha\rho}\Phi_{,\rho} + B_9\epsilon^{\alpha\rho}\Phi_{,\rho}F^{\lambda}_{,\lambda} + B_{10}\epsilon^{\lambda\kappa}\Phi_{,\kappa}F_{,\alpha\lambda} + o(\gamma^3) \tag{5.14b}$$

where

$$B_1 = D_1 + C_1D_3,$$
$$B_2 = D_3(C_4 - Ac_1) - D_2A + a_{10}c_1^2 + 2a_{21}c_1(1 + c_1) + a_{23}c_1^2 + 3a_{18}(1 + c_1)^2$$
$$- D_5(A + D_5) + 2a_{10} + 2c_1(a_{10} + a_{14}) + 3a_3,$$
$$B_3 = c_2D_3 - c_1D_3D_5 - D_2D_5 + a_{10},\ B_4 = D_2c_3 + 2a_{10} + (D_3 - D_2)c_4 + a_{13} + 3a_3,$$
$$B_5 = D_5,\ B_6 = D_5(A + 2D_5) - 2a_{10} - 2c_1(a_{10} + a_{14}) - 6a_3,\ B_7 = 3a_3,$$
$$B_8 = 3a_3,\ B_9 = -2D_6^2 + 2a_{10} + a_{13} + 3a_3 + c_1(2a_{10} + a_{13} + 2a_{14} + a_{17} + 3a_3),$$
$$B_{10} = 2D_6^2 - 3a_3. \tag{5.15}$$

6. DIFFERENTIAL EQUATIONS IN TERMS OF STRESS FUNCTIONS

The compatibility equations expressed in terms of stress functions are derived by substituting (5.14) into (3.6) and (3.7). They are

$$F^{\rho\lambda}_{,\rho\lambda} + M_1F^{\rho}_{,\rho\lambda}F^{\kappa\lambda}_{,\kappa} + M_2F^{\lambda}_{,\rho\kappa}F^{\rho\kappa}_{,\lambda} + M_3F^{\rho}_{,\lambda}F^{\kappa\lambda}_{,\kappa\rho} + M_4\epsilon^{\alpha\kappa}(F^{\rho}_{,\rho\kappa}f_{,\alpha} + F^{\rho}_{,\kappa}f_{,\alpha\rho}) + M_5\Phi^{\rho}_{,\kappa}\Phi^{\kappa}_{,\rho} = o(s^3) \tag{6.1a}$$

$$\Phi^{\rho}_{,\rho} + M_6\Phi_{,\rho}F^{\lambda\rho}_{,\lambda} + M_7\Phi^{\lambda}_{,\rho}F^{\rho}_{,\lambda} = o(s^3), \tag{6.1b}$$

and the auxiliary condition obtained in Section 5 is given by

$$fF^{\rho}_{,\rho} + G^{\rho}_{,\rho} = o(s^3) \tag{6.1c}$$

where

$$M_1 = \frac{1}{B_1}\left(2B_2 - B_7 - 2B_1^2 + B_1B_5 + \frac{1}{2}B_5^2\right),\ M_2 = \frac{1}{B_1}\left(2B_3 + B_7 - \frac{1}{2}B_5^2\right),\ M_3 = \frac{1}{B_1}(2B_3 - B_6),$$
$$M_4 = -\frac{B_5}{B_1},\ M_5 = \frac{1}{B_1}(2B_4 - B_8 - 2D_6^2),\ M_6 = \frac{B_9}{D_6},\ M_7 = -\frac{B_{10}}{D_6}. \qquad (6.2)$$

Any set of functions F, Φ, G, f which satisfies (6.1) and some appropriate boundary conditions is a solution of the generalized plane strain problem. These field equations are different from those obtained by Green and Adkins[5] in the sense that the new stress function such as G is introduced in (6.1).

If we assume that $V^3 = 0$, the stress function Φ becomes zero. In this case, (6.1a) and (6.2a) are reduced to equations for determining the stress field in the sense of usual plane strain, but any mathematical disadvantage may not occur in (6.1).

For a certain class of materials and for a given definite quantity of deformation, there may exist cases where the following relation is satisfied

$$\max\{a_i o^*(\gamma^j)\} = o^*(\gamma),\ j = 3, 4, 5, \ldots$$

where a_i are the material constants which constitute higher order terms than $o(\gamma^2)$ in (4.6), and we have used a symbol "o^*" instead of the conventional symbol "o" in combination with a given definite quantity γ. For such cases some terms involved in error terms in (6.1) should be considered.

7. AN APPLICATION TO AN INFINITE BODY WITH A CIRCULAR HOLE

The theory is illustrated for a circular hole in a body submitted by both uniform anti-plane shear and arbitrary by-axial tension. It is convenient to consider F, Φ and G as functions of the complex variables $z = X^1 + iX^2$ and $\bar{z} = X^1 - iX^2$, $i = \sqrt{-1}$ instead of as functions of X^1 and X^2. In this case the differential equations (6.1) can be rewritten as

$$F_{,zz\bar{z}\bar{z}} + K_1F_{,zz\bar{z}}F_{,z\bar{z}\bar{z}} + K_2F_{,zzz}F_{,\bar{z}\bar{z}\bar{z}} + K_3(F_{,zz}F_{,\bar{z}\bar{z}\bar{z}\bar{z}} + F_{,\bar{z}\bar{z}}F_{,zzzz}) + K_4\Phi_{,zz}\Phi_{,\bar{z}\bar{z}} = o(s^3) \quad (7.1a)$$

$$\Phi_{,z\bar{z}} + K_5(\Phi_{,z}F_{,z\bar{z}\bar{z}} + \Phi_{,\bar{z}}F_{,zz\bar{z}}) + K_6(F_{,zz}\Phi_{,\bar{z}\bar{z}} + F_{,\bar{z}\bar{z}}\Phi_{,zz}) = o(s^3) \quad (7.1b)$$

$$G_{,z\bar{z}} = K_7QF_{,z\bar{z}} + o(s^3) \quad (7.1c)$$

where

$$K_1 = 4M_1 + 3M_2 + 4M_4(A + D_5),\ K_2 = M_2,\ K_3 = M_3 + M_4(A + D_5),$$
$$K_4 = \frac{1}{2}M_5,\ K_5 = 2M_6,\ K_6 = M_7,\ K_7 = A + D_5. \qquad (7.2)$$

Assuming that $\Phi = o(s^2)$, we obtain the field equations for the classical plane strain problem as follows:

$$F_{,zz\bar{z}\bar{z}} + K_1F_{,zz\bar{z}}F_{,z\bar{z}\bar{z}} + K_2F_{,zzz}F_{,\bar{z}\bar{z}\bar{z}} + K_3(F_{,zz}F_{,\bar{z}\bar{z}\bar{z}\bar{z}} + F_{,\bar{z}\bar{z}}F_{,zzzz}) = o(s^3) \quad (7.3a)$$

$$\Phi_{,z\bar{z}} + K_5(\Phi_{,z}F_{,z\bar{z}\bar{z}} + \Phi_{,\bar{z}}F_{,zz\bar{z}}) + K_6(F_{,zz}\Phi_{,\bar{z}\bar{z}} + F_{,\bar{z}\bar{z}}\Phi_{,zz}) = o(s^4) \quad (7.3b)$$

$$G_{,z\bar{z}} = K_7QF_{,z\bar{z}} + o(s^3). \quad (7.3c)$$

On the other hand, if we assume that $F = o(s^2)$, we obtain the field equations for the longitudinal shear problem such that

$$F_{,zz\bar{z}\bar{z}} + K_4\Phi_{,zz}\Phi_{,\bar{z}\bar{z}} = o(s^3) \tag{7.4a}$$

$$\Phi_{,z\bar{z}} = o(s^3) \tag{7.4b}$$

$$G_{,z\bar{z}} = o(s^4). \tag{7.4c}$$

It is found from (7.4c) that in the case of the longitudinal shear problem in-plane stresses $t^{\alpha\beta}$ become symmetric.

For conveniences in obtaining a solution, the stress functions are normalized in the form:

$$F = \sigma_c \overset{(n)}{F}, \Phi = \lambda\sigma_c \overset{(n)}{\Phi}, G = \sigma_c \overset{(n)}{G}, Q = \sigma_c \overset{(n)}{Q} \tag{7.5}$$

where σ_c is defined as the characteristic load and λ denotes arbitrary constant which is responsible for the magnitude of the applied longitudinal shear stress. The equations which govern the normalized stress functions are given by

$$\overset{(n)}{F}_{,zz\bar{z}\bar{z}} + H_1\epsilon \overset{(n)}{F}_{,zz\bar{z}} \overset{(n)}{F}_{,z\bar{z}\bar{z}} + H_2\epsilon \overset{(n)}{F}_{,zzz} \overset{(n)}{F}_{,\bar{z}\bar{z}\bar{z}} + H_3\epsilon(\overset{(n)}{F}_{,zz} \overset{(n)}{F}_{,z\bar{z}\bar{z}\bar{z}} + \overset{(n)}{F}_{,\bar{z}\bar{z}} \overset{(n)}{F}_{,zzz\bar{z}}) + H_4\epsilon \overset{(n)}{\Phi}_{,zz} \overset{(n)}{\Phi}_{,\bar{z}\bar{z}} = o(s^2) \tag{7.6a}$$

$$\overset{(n)}{\Phi}_{,z\bar{z}} + H_5\epsilon(\overset{(n)}{\Phi}_{,z} \overset{(n)}{F}_{,z\bar{z}\bar{z}} + \overset{(n)}{\Phi}_{,\bar{z}} \overset{(n)}{F}_{,zz\bar{z}}) + H_6\epsilon(\overset{(n)}{F}_{,zz} \overset{(n)}{\Phi}_{,\bar{z}\bar{z}} + \overset{(n)}{F}_{,\bar{z}\bar{z}} \overset{(n)}{\Phi}_{,zz}) = o(s^2) \tag{7.6b}$$

$$\overset{(n)}{G}_{,z\bar{z}} = H_7\epsilon \overset{(n)}{Q} \overset{(n)}{F}_{,z\bar{z}} + o(s^2) \tag{7.6c}$$

where

$$H_i = \frac{\sigma_c K_i}{\epsilon}, i = 1, 2, 3, 5, 6, 7, H_4 = \frac{\lambda^2 \sigma_c K_4}{\epsilon} \tag{7.7a}$$

$$\epsilon = \frac{\sigma_c}{K_\sigma} = o(\gamma) \tag{7.7b}$$

where K_σ denotes a reference material constant which we may take as Young's modulus. By the definition the parameter ϵ is found to be of $o(\gamma)$.

Since it is difficult to solve the nonlinear equation (7.6) explicitly, we shall apply the perturbation method. We assume that the stress functions F, G, Φ are functions of the parameter ϵ which can be expanded as absolutely convergent series

$$\overset{(n)}{F} = \sum_{i=0}^{\infty} F_i\epsilon^i, \overset{(n)}{\Phi} = \sum_{i=0}^{\infty} \Phi_i\epsilon^i, \overset{(n)}{G} = \sum_{i=0}^{\infty} G_i\epsilon^i, \overset{(n)}{Q} = \sum_{i=0}^{\infty} Q_i\epsilon^i. \tag{7.8}$$

Now the series (7.8) are substituted into (7.6), then the terms having equal powers of ϵ are collected and finally the coefficients of each power of ϵ are set equal to zero. This process provides a set of linear partial differential equations. The one corresponding to ϵ^0 provides the differential equations for the linear theory. They are

$$F_{0,zz\bar{z}\bar{z}} = 0 \tag{7.9a}$$

$$\Phi_{0,z\bar{z}} = 0 \tag{7.9b}$$

$$G_{0,z\bar{z}} = 0 \tag{7.9c}$$

Since the stress tensor t^{ij} is symmetric in the case of infinitesimal deformation, the function G_0 can be set equal to zero.

From the coefficients of ϵ we have

$$F_{,zz\bar{z}\bar{z}} + H_1 F_{0,zz\bar{z}} F_{0,z\bar{z}\bar{z}} + H_2 F_{0,zzz} F_{0,\bar{z}\bar{z}\bar{z}} + H_3(F_{0,zz} F_{0,z\bar{z}\bar{z}\bar{z}} + F_{0,\bar{z}\bar{z}} F_{0,zzz\bar{z}}) + H_4 \Phi_{0,zz} \Phi_{0,\bar{z}\bar{z}} = 0 \quad (7.10a)$$

$$\Phi_{1,z\bar{z}} + H_5(\Phi_{0,z} F_{0,z\bar{z}\bar{z}} + \Phi_{0,\bar{z}} F_{0,zz\bar{z}}) + H_6(F_{0,zz} \Phi_{0,\bar{z}\bar{z}} + F_{0,\bar{z}\bar{z}} \Phi_{0,zz}) = 0 \quad (7.10b)$$

$$G_{1,z\bar{z}} = H_7 Q_0 F_{0,z\bar{z}}. \quad (7.10c)$$

The differential equations for the higher order terms are derived in a similar way. However the corresponding stresses appear only in the error terms.

For this problem it is also convenient to assume expansions for stress and displacement components, namely

$$t^{\alpha\beta} = \sigma_c \sum_p \overset{(p)}{t}{}^{\alpha\beta} \epsilon^p, \; t^{\alpha 3} = \lambda \sigma_c \sum_p \overset{(p)}{t}{}^{\alpha 3} \epsilon^p, \; P = 0, 1, 2, \ldots \quad (7.11)$$

$$\bar{V}_i = \sum_p \overset{(p)}{V}_i \epsilon^p, \; p = o, 1, 2, \ldots \quad (7.12)$$

Further we introduce a polar coordinate system, where $z = re^{i\theta}$. Let (V_r, V_θ, V_3) be the displacement components expressed in the polar coordinates. Under the circumstances we obtain the expressions of the displacement components in terms of the stress functions as follows:

$$r\bar{V}_{3,r} = 2D_6 i(\Phi_{,z} z - \Phi_{,\bar{z}} \bar{z}) + 4i(2B_9 + B_{10}) F_{,z\bar{z}}(\Phi_{,z} z - \Phi_{,\bar{z}} \bar{z}) + 4iB_{10}(\Phi_{,z} F_{,\bar{z}\bar{z}} \bar{z} - \Phi_{,\bar{z}} F_{,zz} z) + o(\gamma^3) \quad (7.13a)$$

$$\bar{V}_{3,\theta} = -2D_6(\Phi_{,z} z + \Phi_{,\bar{z}} \bar{z}) - 4(2B_9 + B_{10}) F_{,z\bar{z}}(\Phi_{,z} z + \Phi_{,\bar{z}} \bar{z}) + 4B_{10}(\Phi_{,z} F_{,\bar{z}\bar{z}} \bar{z} + \Phi_{,\bar{z}} F_{,zz} z) + o(\gamma^3) \quad (7.13b)$$

$$\begin{aligned} r^2 \bar{V}_{r,r} = {} & 2(2B_1 - B_5) F_{,z\bar{z}} z\bar{z} + 2\{8B_2 + 4B_6 - 4A(A + D_5) + 4B_3 + 2B_7 - D_5^2\} F_{,z\bar{z}} F_{,z\bar{z}} z\bar{z} \\ & + 2(4B_3 + 2B_7 - D_5^2) F_{,zz} F_{,\bar{z}\bar{z}} z\bar{z} + 2(2B_4 - 2D_6^2 - B_8) \Phi_{,z} \Phi_{,\bar{z}} z\bar{z} - B_5(F_{,zz} z^2 + F_{,\bar{z}\bar{z}} \bar{z}^2) \\ & + 2\{2B_6 + 2D_5(A + D_5) + 2B_7 - D_5^2\} F_{,z\bar{z}}(F_{,zz} z^2 + F_{,\bar{z}\bar{z}} \bar{z}^2) + (2D_6^2 - B_8) \\ & \times (\Phi_{,z} \Phi_{,z} z^2 + \Phi_{,\bar{z}} \Phi_{,\bar{z}} \bar{z}^2) + B_5 i(G_{,zz} z^2 - G_{,\bar{z}\bar{z}} \bar{z}^2) + o(\gamma^3) \end{aligned} \quad (7.13c)$$

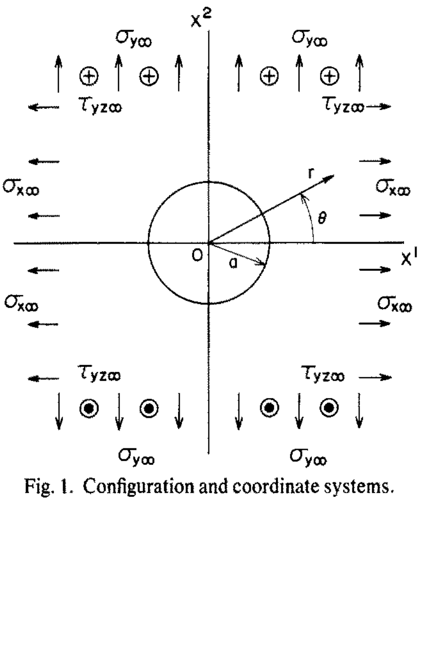

Fig. 1. Configuration and coordinate systems.

$$r^3(\bar{V}_r + \bar{V}_{\theta,\theta}) = 2(2B_1 - B_5)F_{,z\bar{z}}z\bar{z} + 2\{8B_2 + 4B_6 - 4A(A + D_5) + 4B_3 + 2B_7 - D_5^2\}F_{,z\bar{z}}F_{,z\bar{z}}z\bar{z}$$
$$+ 2(4B_3 + 2B_7 - D_5^2)\, F_{,zz}F_{,\bar{z}\bar{z}}z\bar{z} + 2(2B_4 - 2D_6^2 - B_8)\Phi_{,z}\Phi_{,\bar{z}}z\bar{z} + B_5(F_{,zz}z^2 + F_{,\bar{z}\bar{z}}\bar{z}^2)$$
$$- 2\{2B_6 + 2D_5(A + D_5) + 2B_7 - D_5^2\}\, F_{,z\bar{z}}(F_{,zz}z^2 + F_{,\bar{z}\bar{z}}\bar{z}^2) - (2D_6^2 - B_8)\,(\Phi_{,z}\Phi_{,z}z^2 + \Phi_{,\bar{z}}\Phi_{,\bar{z}}\bar{z}^2)$$
$$- B_5 i(G_{,zz}z^2 - G_{,\bar{z}\bar{z}}\bar{z}^2) + o(\gamma^3) \quad (7.13d)$$

$$r^3\left(\bar{V}_{r,\theta} + \frac{\bar{V}_{\theta,\lambda}}{r} - \bar{V}_\theta\right) = -2B_5 i(F_{,zz}z^2 - F_{,\bar{z}\bar{z}}\bar{z}^2)$$
$$+ 4i\{2B_6 + 2D_5(A + D_5) + 2B_7 - D_5^2\}\, F_{,z\bar{z}}(F_{,zz}z^2 - F_{,\bar{z}\bar{z}}\bar{z}^2) + 2i(2D_6 - B_8)\,(\Phi_{,z}\Phi_{,z}z^2 - \Phi_{,\bar{z}}\Phi_{,\bar{z}}\bar{z}^2)$$
$$- 2B_5(G_{,zz}z^2 + G_{,\bar{z}\bar{z}}\bar{z}^2) + o(\gamma^3). \quad (7.13e)$$

It should be noted that the displacement components must be single-valued functions of their supports.

The stress functions must satisfy the boundary conditions. The stress free conditions on the rim of the hole, i.e. on the circle $r = a$, are

$$F_{0,zz}z^2 = F_{0,\bar{z}\bar{z}}\bar{z}^2 = F_{0,z\bar{z}}z\bar{z} \quad (7.14a)$$

$$2F_{k,z\bar{z}}z\bar{z} - F_{k,zz}z^2 - F_{k,\bar{z}\bar{z}}\bar{z}^2 + i(G_{k,zz}z^2 - G_{k,\bar{z}\bar{z}}\bar{z}^2) = 0,\ k \geq 1 \quad (7.14b)$$

$$-i(F_{k,zz}z^2 - F_{k,\bar{z}\bar{z}}\bar{z}^2) - G_{k,zz}z^2 + 2G_{k,z\bar{z}}z\bar{z} - G_{k,\bar{z}\bar{z}}\bar{z}^2 = 0,\ k \geq 1 \quad (7.14c)$$

$$\Phi_{k,z}z = \Phi_{k,\bar{z}}\bar{z},\ k \geq 0. \quad (7.14d)$$

The conditions at infinity are written by

$$F_{0,11} = \kappa_1,\ F_{0,22} = \kappa_2,\ F_{0,12} = 0 \quad (7.14e)$$

$$F_{k,22} + G_{k,12} = 0,\ F_{k,11} - G_{k,12} = 0,\ -F_{k,12} + G_{k,22} = 0,\ F_{k,12} + G_{k,11} = 0,\ k \geq 1 \quad (7.14f)$$

$$\Phi_{k,2} = \Phi_{k,1} = 0,\ k \geq 1 \quad (7.14g)$$

$$\Phi_{0,1} = -\kappa_3,\ \Phi_{0,2} = 0 \quad (7.14h)$$

where

$$\kappa_1 = \frac{\sigma_{y\infty}}{\sigma_c},\ \kappa_2 = \frac{\sigma_{x\infty}}{\sigma_c},\ \kappa_3 = \frac{\tau_{yz\infty}}{\lambda\sigma_c} \quad (7.15)$$

where $\sigma_{x\infty}$, $\sigma_{y\infty}$ and $\tau_{yz\infty}$ mean the intensity of the uniformly distributed stresses t^{11}, t^{22} and t^{23} at $r \to \infty$ respectively.

The solution based on the linear theory is derived from (7.9) with the boundary conditions (7.14a, e, h) as follows:

$$F_0 = -a^2\gamma_1\left(\frac{\bar{z}}{z} + \frac{z}{\bar{z}}\right) + \gamma_2 z\bar{z} + \frac{\gamma_1}{2}(z^2 + \bar{z}^2) - a^2\gamma_2 \log z\bar{z} + \frac{a^4}{2}\gamma_1\left(\frac{1}{z^2} + \frac{1}{\bar{z}^2}\right) \quad (7.16a)$$

$$Q_0 = 4a^2\gamma_1 i\left(\frac{1}{\bar{z}^2} - \frac{1}{z^2}\right) \quad (7.16b)$$

$$\Phi_0 = -\frac{\kappa_3}{2}\left(z + \bar{z} - \frac{a^2}{\bar{z}} - \frac{a^2}{z}\right) \quad (7.17)$$

where

$$\gamma_1 = \frac{\kappa_1 - \kappa_2}{4}, \quad \gamma_2 = \frac{\kappa_1 + \kappa_2}{4}. \quad (7.18)$$

Substituting (7.16), (7.17) into (7.10), we obtain

$$F_{,zz\bar{z}\bar{z}}+\frac{a^4}{z^3\bar{z}^3}N_1+\frac{a^6}{z^4\bar{z}^4}N_2+\frac{a^8}{z^5\bar{z}^5}N_3+a^4\left(\frac{1}{z^2\bar{z}^4}+\frac{1}{z^4\bar{z}^2}\right)N_4+a^6\left(\frac{1}{z^3\bar{z}^5}+\frac{1}{z^5\bar{z}^3}\right)N_5$$
$$+a^2\left(\frac{1}{z^4}+\frac{1}{\bar{z}^4}\right)N_6=0 \quad (7.19a)$$

$$\Phi_{1,z\bar{z}}+a^2\left(\frac{1}{z^3}+\frac{1}{\bar{z}^3}\right)N_7+a^4\left(\frac{1}{z^2\bar{z}^3}+\frac{1}{\bar{z}^2z^3}\right)N_8+a^6\left(\frac{1}{z^4\bar{z}^3}+\frac{1}{\bar{z}^4z^3}\right)N_9=0 \quad (7.19b)$$

$$G_{1,z\bar{z}}=a^4i\left(\frac{1}{\bar{z}^4}-\frac{1}{z^4}\right)N_{10}+a^2i\left(\frac{1}{\bar{z}^2}-\frac{1}{z^2}\right)N_{11} \quad (7.19c)$$

where we have put

$$N_1=4H_1\gamma_1^2+4H_2(\gamma_2^2+9\gamma_1^2)-24\gamma_1^2H_3+\kappa_3^2H_4,\ N_2=\gamma_1^2(18H_3-144H_2)$$
$$N_3=144\gamma_1^2H_2,\ N_4=6\gamma_1\gamma_2(H_3-2H_2),\ N_5=24\gamma_1\gamma_2H_2$$
$$N_6=6\gamma_1^2H_3,\ N_8=\kappa_3(\gamma_1H_5-2\gamma_1H_6+\gamma_2H_6),\ N_7=\kappa_3\gamma_1(H_5+H_6),$$
$$N_9=3\kappa_3\gamma_1H_6,\ N_{10}=4H_7\gamma_1^2,\ N_{11}=4H_7\gamma_1\gamma_2. \quad (7.20)$$

Considering the fact that there is some freedom in the choice of the stress functions F_1, G_1, Φ_1 and the condition that displacement components should be single-valued functions of their supports, we obtain the solution for (7.19) with boundary conditions (7.14b, c, d, f, g) such that

$$F_1=-\frac{N_1a^4}{4}\frac{1}{z\bar{z}}-\frac{N_2a^6}{36}\frac{1}{z^2\bar{z}^2}-\frac{N_3a^8}{144}\frac{1}{z^3\bar{z}^3}+\frac{N_4a^4}{6}\left(\frac{1}{z^2}+\frac{1}{\bar{z}^2}\right)(\log z\bar{z}-\log a^2)$$
$$+\left(\frac{5N_4}{36}a^4+\frac{P_4}{6}\right)\left(\frac{1}{z^2}+\frac{1}{\bar{z}^2}\right)-\frac{N_5a^6}{24}\left(\frac{1}{z\bar{z}^3}+\frac{1}{z^3\bar{z}}\right)-\frac{N_6a^2}{12}\left(\frac{\bar{z}^2}{z^2}+\frac{z^2}{\bar{z}^2}\right)$$
$$+\frac{R_3}{2}\left(\frac{\bar{z}}{z}+\frac{z}{\bar{z}}\right)+\frac{R_5}{12}\left(\frac{\bar{z}}{z^3}+\frac{z}{\bar{z}^3}\right)-P_2\log z\bar{z}+\frac{P_6}{20}\left(\frac{1}{z^4}+\frac{1}{\bar{z}^4}\right) \quad (7.21a)$$

$$G_1=\frac{a^4N_{10}}{3}i\left(\frac{\bar{z}}{z^3}-\frac{z}{\bar{z}^3}\right)+N_{11}a^2i\left(\frac{\bar{z}}{z}-\frac{z}{\bar{z}}\right) \quad (7.21b)$$

$$\Phi_1=\frac{N_7}{2}a^2\left(\frac{\bar{z}}{z^2}+\frac{z}{\bar{z}^2}\right)-\frac{N_8a^4}{2}\left(\frac{1}{z\bar{z}^2}+\frac{1}{\bar{z}z^2}\right)-\frac{N_9a^6}{6}\left(\frac{1}{z^3\bar{z}^2}+\frac{1}{\bar{z}^3z^2}\right)-T_2\left(\frac{1}{z}+\frac{1}{\bar{z}}\right)-\frac{T_4}{3}\left(\frac{1}{z^3}+\frac{1}{\bar{z}^3}\right) \quad (7.21c)$$

where

$$R_3=\left(N_{11}-\frac{N_4}{6}-\frac{N_5}{24}\right)a^2,\ R_5=2a^4(N_6+2N_{10}),$$
$$P_2=\left(\frac{N_1}{4}+\frac{N_2}{18}+\frac{N_3}{48}\right)a^2,\ P_4=\left(-\frac{N_4}{12}+\frac{7}{16}N_5-\frac{9}{2}N_{11}\right)a^4,\ P_6=-5a^6\left(\frac{N_6}{3}+2N_{10}\right),$$
$$T_2=-\frac{a^2}{2}\left(N_8+\frac{N_9}{3}\right),\ T_4=\frac{3}{2}N_7a^4. \quad (7.22)$$

The physical stress components defined in the cylindrical coordinate system are expressed in terms of the stress functions as follows:

$$r^2t^{rr} = 2F_{,z\bar{z}}z\bar{z} - F_{,zz}z^2 - F_{,\bar{z}\bar{z}}\bar{z}^2 + i(G_{,zz}z^2 - G_{,\bar{z}}\bar{z}^2) \tag{7.23a}$$

$$r^4t^{\theta\theta} = 2F_{,z\bar{z}}z\bar{z} + F_{,zz}z^2 + F_{,\bar{z}\bar{z}}\bar{z}^2 - i(G_{,zz}z^2 - G_{,\bar{z}\bar{z}}\bar{z}^2) \tag{7.23b}$$

$$r^3t^{r\theta} = -i(F_{,zz}z^2 - F_{,\bar{z}\bar{z}}\bar{z}^2) + 2G_{,z\bar{z}}z\bar{z} - G_{,zz}z^2 - G_{,\bar{z}\bar{z}}\bar{z}^2 \tag{7.23c}$$

$$r^3t^{\theta r} = -i(F_{,zz}z^2 - F_{,\bar{z}\bar{z}}\bar{z}^2) - 2G_{,z\bar{z}}z\bar{z} - (G_{,zz}z^2 + G_{,\bar{z}\bar{z}}\bar{z}^2) \tag{7.23d}$$

$$r^2t^{\theta 3} = -\Phi_{,z}z - \Phi_{,\bar{z}}\bar{z},\ rt^{r3} = i(\Phi_{,z}z - \Phi_{,\bar{z}}\bar{z}) \tag{7.23e}$$

$$rt^{3r} = i\{1 + (4q_1 + 2q_2)F_{,z\bar{z}}\}(\Phi_{,z}z - \Phi_{,\bar{z}}\bar{z}) + 2q_2 i(\Phi_{,z}F_{,\bar{z}\bar{z}}\bar{z} - \Phi_{,\bar{z}}F_{,zz}z) \tag{7.23f}$$

$$r^2t^{3\theta} = -\{1 + (4q_1 + 2q_2)F_{,z\bar{z}}\}(\Phi_{,z}z + \Phi_{,z}z) + 2q_2(\Phi_{,z}F_{,\bar{z}\bar{z}}\bar{z} + \Phi_{,\bar{z}}F_{,zz}z) \tag{7.23g}$$

$$t^{33} = 4c_1F_{,z\bar{z}} + q_3F_{,z\bar{z}}F_{,z\bar{z}} + 8(c_2 - c_1D_5)F_{,zz}F_{,\bar{z}\bar{z}} + 4(c_3 + 2D_6)\Phi_{,z}\Phi_{,\bar{z}} \tag{7.23h}$$

where

$$q_1 = A + D_5 - 2D_6,\ q_2 = 2D_6 - D_5,\ q_3 = 8\{c_2 - c_1D_5 + 2(c_4 - Ac_1)\}. \tag{7.24}$$

The stress distribution on the rim of the hole is of special interest for engineers. The physical stress components on the circle $r = a$ are given by

$$\begin{aligned}\frac{a^2t^{\theta\theta}}{\sigma_c} = 4\gamma_2 + 8\gamma_1\cos 2\theta + \frac{2\epsilon}{a^2}\{\pi_1 + \pi_2 + (2\pi_2 - \pi_5 - \pi_6 + 2N_{11}a^2)\cos 2\theta \\ + (2\pi_3 - \pi_7 - \pi_8 + 4N_{10}a^2)\cos 4\theta\} + o(\gamma^2)\end{aligned} \tag{7.25a}$$

$$\frac{at^{\theta r}}{\sigma_c} = 8\epsilon(N_{11}\sin 2\theta + N_{10}\sin 4\theta) + o(\gamma^2) \tag{7.25b}$$

$$\frac{at^{\theta 3}}{\lambda\sigma_c} = 2\cos\theta - \frac{4\epsilon}{a}(\pi_{10}\cos\theta + \pi_9\cos 3\theta) + o(\gamma^2) \tag{7.25c}$$

$$\frac{at^{3\theta}}{\lambda\sigma_c} = 2\cos\theta + \epsilon\left\{8q_1k_\sigma(2\gamma_1\cos 2\theta + \gamma_2)\cos\theta - \frac{4}{a}(\pi_9\cos 3\theta + \pi_{10}\cos\theta)\right\} + o(\gamma^2) \tag{7.25d}$$

$$\frac{t^{3\gamma}}{\sigma_c} = o(\gamma^2) \tag{7.25e}$$

$$\begin{aligned}\frac{t^{33}}{\sigma_c} = 4c_1(\gamma_2 + 2\gamma_1\cos 2\theta) + 4\epsilon\left\{\frac{c_1}{a^2}(\pi_1 + 2\pi_2\cos 2\theta + 2\pi_3\cos 4\theta)\right. \\ \left. + (q_4 + q_5)(\gamma_2 + 2\gamma_1\cos 2\theta)^2 + \lambda^2 q_6\cos^2\theta\right\} + o(\gamma^2)\end{aligned} \tag{7.25f}$$

where

$$\pi_1 = -\left(\frac{N_1}{4} + \frac{N_2}{9} + \frac{N_3}{16}\right)a^2,\ \pi_2 = -\left(\frac{N_{11}}{2} + \frac{N_4}{4} + \frac{5}{8}N_5\right)a^2,\ \pi_3 = -\left(\frac{N_6}{6} + N_{10}\right)a^2,$$

$$\pi_4 = -\left(\frac{N_1}{4} + \frac{N_2}{9} + \frac{N_3}{16}\right)a^2,\ \pi_5 = \left(\frac{N_4}{6} + \frac{N_5}{12}\right)a^2,\ \pi_6 = \left(\frac{5}{48}N_5 + \frac{N_4}{4} + \frac{7}{2}N_{11}\right)a^2,$$

$$\pi_7 = \left(\frac{N_6}{6} + 6N_{10}\right)a^2,\ \pi_8 = \frac{N_6}{6}a^2,\ \pi_9 = \frac{N_7}{2}a,\ \pi_{10} = \left(\frac{N_8}{2} + \frac{N_9}{3}\right)a,\ q_4 = 2K_\sigma\{c_2 - c_1D_5 + 2(c_4 + Ac_1)\},$$

$$q_5 = 2K_\sigma(c_2 - c_1D_5),\ q_6 = K_\sigma(c_3 + 2D_6). \tag{7.26}$$

In the course of obtaining (7.25), we have put $\kappa_3 = 1$.

The functions F_n, G_n, Φ_n, $n \geqq 2$ can be formally solved in like manner. However, the corresponding stresses appear only in the error terms in (7.25). If it is required to obtain the higher order stresses, the more refined field equations should be employed instead of (6.1).

8. NUMERICAL ILLUSTRATION

As a special case of Section 8, we can consider the case where the magnitude of the applied stress $t_{y\infty}$ is equal to that of $t_{zy\infty}$ and the applied stress $t_{x\infty}$ is zero. In this case we can put $\kappa_1 = 1$, $\kappa_2 = 0$, $\lambda = 1$ and define the characteristic load as $t_{y\infty}$ or $t_{zy\infty}$.

Next we consider how to determine the material constants which appear in nonlinear constitutive equations. In the following the material constants which constitute the nonlinear terms in stress–strain relations shall simply be called the nonlinear material constants. Generally speaking, these material constants should be determined from tension test, shear test, etc. Since a set of appropriate experimental data is not at hand, we may be obliged to invent such conditions that the nonlinear material constants can be determined for illustration purposes.

If the deformation is small, it can be assumed from the variation of the complementary energy function we obtain

$$\gamma_{ij} = E_{ijkl}s^{kl} + \frac{1}{2}E_{ijklmn}s^{kl}s^{mn} + o(s^3) \tag{8.1}$$

where E_{ijkl} is the linear material tensor and E_{ijklmn} is the nonlinear material tensor. These material tensors have the following symmetry relations:

$$E_{ijkl} = E_{(ij)(kl)} = E_{klij} \tag{8.2a}$$

$$E_{ijklmn} = E_{(ij)(kl)(mn)} = E_{klijmn} = E_{ijmnkl} = E_{mnklij}. \tag{8.2b}$$

Let us suppose that the tensor E_{ijklmn} is decomposed with respect to the pair of tensor fields E_{ijkl}, f_{ij}. Here f_{ij} is a symmetric tensor which may later be determined. It is easily seen that this decomposition leads to

$$\begin{aligned} E_{ijklmn}(\alpha, \beta, \gamma) = {} & \alpha(E_{ijkl}f_{mn} + E_{klmn}f_{ij} + E_{mnij}f_{kl}) + \beta(E_{ijkm}f_{nl} + E_{kmnl}f_{ij} + E_{nlij}f_{km} + E_{ijkn}f_{lm} + E_{knlm}f_{ij} \\ & + E_{lmij}f_{kn} + E_{jkil}f_{mn} + E_{ilmn}f_{jk} + E_{mnjk}f_{il} + E_{kijl}f_{mn} + E_{jlmn}f_{ki} + E_{mnki}f_{jl} + E_{klim}f_{jn} \\ & + E_{imjn}f_{kl} + E_{jnkl}f_{im} + E_{klin}f_{jm} + E_{injm}f_{kl} + E_{jmkl}f_{in}) + \gamma(E_{jkim}f_{nl} + E_{imnl}f_{kj} + E_{nlkj}f_{im} \\ & + E_{jkin}f_{lm} + E_{inlm}f_{jk} + E_{lmjk}f_{in} + E_{kijm}f_{nl} + E_{jmnl}f_{ki} + E_{nlki}f_{jm} + E_{kijn}f_{lm} + E_{jnlm}f_{ki} \\ & + E_{lmki}f_{jn} + E_{iljm}f_{kn} + E_{jmkn}f_{il} + E_{knil}f_{jm} + E_{jlkm}f_{in} + E_{kmin}f_{il} + E_{injl}f_{km} + E_{iljn}f_{km} \\ & + E_{jnkm}f_{il} + E_{kmil}f_{jn} + E_{jlkn}f_{im} + E_{knim}f_{jl} + E_{imjl}f_{kn}) \end{aligned} \tag{8.3}$$

where

$$f_{ij} = f_{(ij)}. \tag{8.4}$$

In the case of transverse isotropy, we may write down the following suppositions.

Supposition 1. In the case of $i, j, k, l, m, n < 3$, the nonlinear material tensor E_{ijklmn} is given by

$$E_{ijklmn} = E_{ijklmn}(\alpha, \beta, \beta). \tag{8.5a}$$

In other cases, this tensor is

$$E_{ijklmn} = E_{ijklmn}(\alpha_0, \beta_0, \beta_0) \tag{8.5b}$$

where α_0, β_0, α and β are the constants which may be determined later.

Supposition 1 is the necessary condition for the following equations to hold;

$$E_{111133} = E_{222233}, E_{331313} = E_{332323}, E_{333311} = E_{333322}, E_{111212} = E_{221212}, E_{111122} = E_{112222},$$
$$E_{111313} - E_{112323} = 2E_{131223}, E_{111133} - E_{112233} = 2E_{331212}, E_{111111} = E_{111122} + 4E_{111111}. \tag{8.6}$$

Supposition 2. The symmetric tensor f_{ij} is given by

$$f_{ij} = E_{1111}(\delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2}) + E_{3333}\delta_{i3}\delta_{j3}. \tag{8.7}$$

Such components of the tensor E_{ijklmn} as E_{121212} must be equal to zero in the case of transverse isotropy. This requires Supposition 2.

In the case of uniaxial tension, we find from (8.1) that the ratio of the lateral contraction to the negative of the longitudinal extension becomes

$$\nu_1'(\sigma) = -\frac{E_{2211}\sigma + \frac{1}{2}E_{221111}\sigma^2}{E_{1111}\sigma + \frac{1}{2}E_{111111}\sigma^2} \tag{8.8a}$$

where σ is the magnitude of the tensile stress. Similarly we have

$$\nu_2'(\sigma) = -\frac{E_{3311}\sigma + \frac{1}{2}E_{331111}\sigma^2}{E_{3333}\sigma + \frac{1}{2}E_{333333}\sigma^2}. \tag{8.8b}$$

The ratios $\nu_\alpha'(\sigma)$, $\alpha = 1, 2$ reduce to the classical values of Poisson's ratios ν_α when the tensile stress σ is sufficiently small.

If σ is fairly large and if the magnitude of σ is sufficient for the yield strength, the ratios ν_α' may be considered to reduce to 0·5.

In this section, however, we confine our attention to the case where the ratios $\nu_\alpha'(\sigma)$ are assumed to be independent of the tensile stress σ within the error of $o(\gamma^3)$. If the second order terms are very small compared with the first order terms in (8.8), the ratios $\nu_\alpha'(\sigma)$ are rewritten in the form:

$$\nu_1'(\sigma) = \nu_1\left\{1 + \frac{1}{2}\left(\frac{E_{221111}}{E_{2211}} - \frac{E_{111111}}{E_{1111}}\right)\sigma\right\} + o(\sigma^2),\ \nu_2'(\sigma) = \nu_2\left\{1 + \frac{1}{2}\left(\frac{E_{331111}}{E_{3311}} - \frac{E_{333333}}{E_{3333}}\right)\sigma\right\} + o(\sigma^2). \tag{8.9}$$

If the ratios ν_α' are independent of the stress σ under the circumstances the following equations are obtained:

$$\frac{E_{221111}}{E_{111111}} = -\nu_1 \tag{8.10a}$$

$$\frac{E_{331111}}{E_{333333}} = -\nu_2. \tag{8.10b}$$

Table 1. Linear elastic constants† (10^{-7} cm²/kg)

D_1	D_2	D_3	D_4	D_5	D_6
4·902	*−1.471*	*−1.428*	5·519	*6·373*	*6·898*

†The italic numbers are assumed values.

Table 2. Linear elastic constants (10^{-7} cm²/kg)

D_1	D_2	D_3	D_4	D_5	D_6
4·902	−1·471	−1·471	4·902	6·373	6·373

Supposition 3. When the deformation is relatively small, the equations (8.10) hold within the error of $o(\gamma^3)$.

If the material is incompressible under hydrostatic loading, Supposition 3 is in good agreement with the result reported by Orthwein[7].

With the aid of Supposition 1–3, we can determine such constants as α, β, α_0, β_0. That is, they are determined from the following equations:

$$3(\alpha + 14\beta)E_{1111}f_{11} = E_{111111} \tag{8.11a}$$

$$\alpha(E_{1111} + 2E_{1122})f_{11} + 2\beta(E_{1111} + 6E_{1212} + 2E_{1122})f_{11} = -\nu_1 E_{111111} \tag{8.11b}$$

$$3(\alpha_0 + 14\beta_0)E_{3333}f_{33} = E_{333333} \tag{8.11c}$$

$$\alpha_0(E_{1111}f_{33} + 2E_{1133}f_{11}) + 2\beta_0(E_{1111}f_{33} + 6E_{1313}f_{11} + 2E_{1133}f_{11}) = -\nu_2 E_{333333} \tag{8.11d}$$

In particular, the nonlinear material tensors E_{111111} and E_{333333} are directly determined from stress–strain curves in the directions parallel to X^1 and X^3 axes respectively.

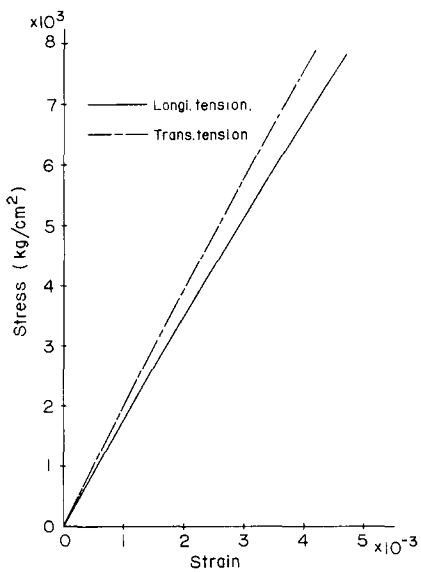

Fig. 2. Stress–strain diagrams for Cr–Ni steel.

Then the nonlinear material constants are given by

$$\begin{aligned}
&3a_3 = 4E_{122331},\ 2a_{10} = 2E_{111212} - 4E_{122331},\ 6a_{18} = E_{111222} - 2E_{111212} + 4E_{122331},\\
&2a_{21} = E_{112233} - E_{111122} + 2E_{111212} - 4E_{122331},\ a_{13} = 2E_{111313} - 2E_{111212},\\
&2a_{23} = E_{11333} - 4E_{111212} - 2E_{112233} + 2E_{111122} + 8E_{122331},\\
&2a_{14} = E_{111133} - 2E_{111212} - E_{112233} + 4E_{122331},\\
&a_{17} = 2E_{331313} - E_{111133} + 2E_{111212} + E_{112233} - 4E_{122331} - 2E_{111313},\\
&6a_{19} = E_{333333} - 3E_{111133} - E_{112211} + 8E_{111212} - 3E_{113333} + 6E_{112233} - 24E_{122331}.
\end{aligned} \tag{8.12}$$

In order to carry out some numerical calculations, let us use the experimental data on Cr–Ni steel [9].

Since these data are varied for the plate which have a thickness of 0·051 cm, there are some problems in applying these data to the present investigation. Here we don't consider the difference between the material constants for design and those used in the present illustrations, because the aim of some numerical calculations is to show the second order effect on the stress concentration factor qualitatively.

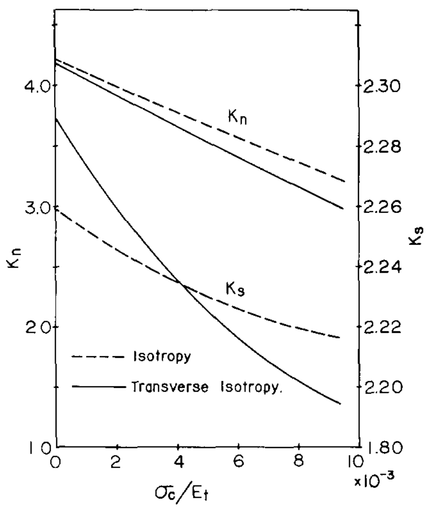

Fig. 3. Stress concentration factors K_n, K_s.

The stress–strain curves for Cr–Ni steel may be approximated in the form:

$$\gamma = \frac{\sigma}{E_l} + 19{\cdot}68\left(\frac{\sigma}{E_l}\right)^2 + o(\gamma^3)\ \text{for longitudinal tension} \tag{8.13a}$$

$$\gamma = \frac{\sigma}{E_t} + 19{\cdot}66\left(\frac{\sigma}{E_t}\right)^2 + o(\gamma^3)\ \text{for transverse tension} \tag{8.13b}$$

where σ is nominal stress, γ is strain and E_l and E_t are Young's moduli in the longitudinal and transverse directions respectively.

If transverse and longitudinal directions are taken as directions parallel to X^1 and X^3 axes respectively, the components E_{111111}, E_{333333} of the nonlinear material tensor E_{ijklmn} are given by

$$E_{111111} = \frac{2 \times 19{\cdot}66}{E_t^2}, \quad E_{333333} = \frac{2 \times 19{\cdot}68}{E_e^2}. \tag{8.14}$$

In the remainder of the discussion, K_σ shall be defined as E_t, and we shall assume that the elastic properties of Cr–Ni steel are given by Table 1.

The stress–strain curves approximated by (8.13) are illustrated in Fig. 2.

Let us define the stress concentration factors K_n, K_s

$$K_n = \frac{\sigma_{max}}{\sigma_{y\infty}}, \quad K_s = \frac{\tau_{max}}{\tau_{yz\infty}} \tag{8.15}$$

where σ_{max} and τ_{max} are the maximum values of normal stress and of longitudinal shear stress respectively. The relation between these factors and the applied stress σ_c/E_t is shown in Fig. 3.

In order to investigate the effect of transverse isotropy on the stress distributions on $r = a$, we shall consider an assumption material which may be assumed to be isotropic and to have the stress–strain curve given by (8.13b) and the elastic properties listed in Table 2. Under the circumstances, the comparison between the isotropic material and the transversely isotropic materials is also shown in Fig. 3. It is found from Fig. 3 that stress concentration factors decrease monotonously as the nondimensional applied stresses increase, and that these factors decrease more rapidly in the case of transverse isotropy than in the case of isotropy.

The longitudinal shear stress becomes maximum on the plane which is generated by rotating the X^3 = constant plane through an angle α about the axis X^1. This value of α is plotted against σ_c/E_t in Fig. 4. The corresponding maximum stress occurs at $(a, 0)$ on the plane inclined at $\alpha + \pi/4$ from the X^3 = constant plane.

In Fig. 5 stress distributions on $r = a$ are illustrated for $\sigma_c/E_t = 0{\cdot}005$. In particular the stress component $t^{\theta r}$ is the second order stress, and so $t^{\theta r}$ is considered to be nonlinear effect. It is interesting to note that the effect of the second order stress on the stress distributions on $r = a$ is

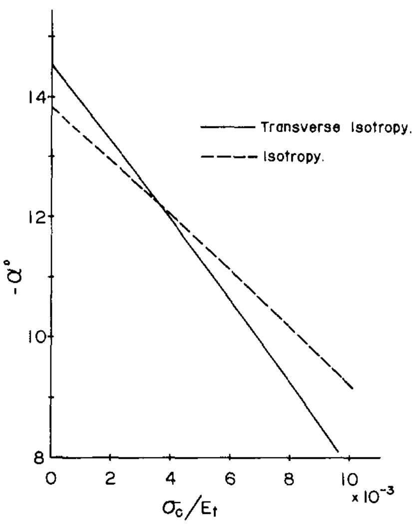

Fig. 4. The value of α in degree.

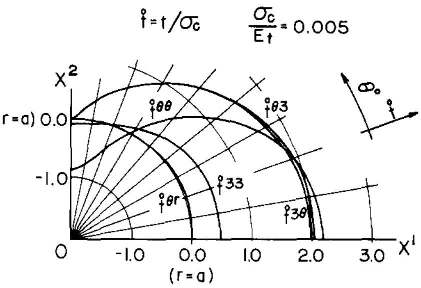

Fig. 5. Stress distributions on $r = a$ for $\sigma_c/E_t = 0{\cdot}005$.

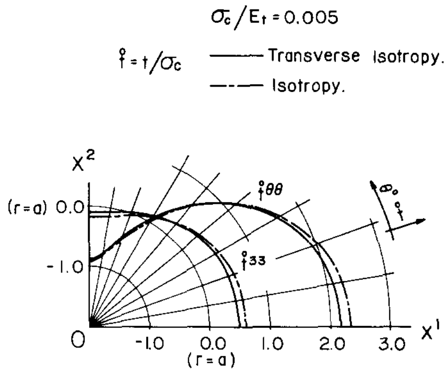

Fig. 6. The effect of transverse isotropy on the stress distributions on $r = a$.

fairly large, and so such stress as $t^{\theta r}$ can not be necessarily neglected in the nonlinear plane strain problems.

The effect of transverse isotropy on the stress distributions of $t^{\theta\theta}$ and t^{33} on $r = a$ is shown in Fig. 6. It may be observed from Fig. 6 that in this case the effect of anisotropy on the stress distributions on $r = a$ is fairly large.

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